

# DECAY ESTIMATES OF A TANGENTIAL DERIVATIVE TO THE LIGHT CONE FOR THE WAVE EQUATION AND THEIR APPLICATION

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**ABSTRACT.** We consider wave equations in three space dimensions and obtain new weighted  $L^\infty$ - $L^\infty$  estimates for a tangential derivative to the light cone. As an application, we give a new proof of the global existence theorem, which was originally proved by Klainerman and Christodoulou, for systems of nonlinear wave equations under the null condition. Our new proof has the advantage of using neither the scaling nor the Lorentz boost operators.

## 1. INTRODUCTION

Solutions to the Cauchy problem for nonlinear wave equations with quadratic nonlinearity in three space dimensions may blow up in finite time no matter how small initial data are, and we have to impose some special condition on the nonlinearity to get global solutions. The null condition is one of such conditions and is associated with the null forms  $Q_0$  and  $Q_{ab}$ , which are given by

$$(1.1) \quad Q_0(v, w; c) = (\partial_t v)(\partial_t w) - c^2 (\nabla_x v) \cdot (\nabla_x w),$$

$$(1.2) \quad Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w) \quad (0 \leq a < b \leq 3)$$

for  $v = v(t, x)$  and  $w = w(t, x)$ , where  $c$  is a positive constant corresponding to the propagation speed,  $\partial_0 = \partial_t = \partial/\partial t$ , and  $\partial_j = \partial/\partial x_j$  ( $j = 1, 2, 3$ ). More precisely, let  $c > 0$  and consider the Cauchy problem for

$$(1.3) \quad \square_c u_i = F_i(u, \partial u, \nabla_x \partial u) \quad \text{in } (0, \infty) \times \mathbb{R}^3 \quad (1 \leq i \leq m)$$

with initial data

$$(1.4) \quad u = \varepsilon f \text{ and } \partial_t u = \varepsilon g \quad \text{at } t = 0,$$

where  $\square_c = \partial_t^2 - c^2 \Delta_x$ ,  $u = (u_j)$ ,  $\partial u = (\partial_a u_j)$ , and  $\nabla_x \partial u = (\partial_k \partial_a u_j)$  with  $1 \leq j \leq m$ ,  $1 \leq k \leq 3$ , and  $0 \leq a \leq 3$ , while  $\varepsilon$  is a positive parameter. Let  $F = (F_i)_{1 \leq i \leq m}$  be quadratic around the origin in its arguments and the system be quasi-linear. In other words, we assume that each  $F_i$  has the form

$$(1.5) \quad F_i(u, \partial u, \nabla_x \partial u) = \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq 3, \ 0 \leq a \leq 3}} c_{ka}^{ij}(u, \partial u) \partial_k \partial_a u_j + d_i(u, \partial u),$$

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where  $c_{ka}^{ij}(u, \partial u) = O(|u| + |\partial u|)$  and  $d_i(u, \partial u) = O(|u|^2 + |\partial u|^2)$  around  $(u, \partial u) = (0, 0)$ . Without loss of generality, we may assume  $c_{k\ell}^{ij} = c_{\ell k}^{ij}$  for  $1 \leq i, j \leq m$  and  $1 \leq k, \ell \leq 3$ . In addition, we always assume the symmetry condition

$$c_{ka}^{ij} = c_{ka}^{ji} \quad \text{for } 1 \leq i, j \leq m, \quad 1 \leq k \leq 3, \text{ and } 0 \leq a \leq 3.$$

Then it is well known that the null condition (for the above system (1.3)) is satisfied if and only if the quadratic terms of  $F_i$  ( $1 \leq i \leq m$ ) can be written as linear combinations of the null forms  $Q_0(u_j, \partial^\alpha u_k; c)$  and  $Q_{ab}(u_j, \partial^\alpha u_k)$  with  $1 \leq j, k \leq m$ ,  $0 \leq a < b \leq 3$ , and  $|\alpha| \leq 1$ , where  $\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$  for a multi-index  $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  (refer to [3] and [14] for the precise description of the null condition). Klainerman [14] and Christodoulou [3] proved the following global existence theorem independently by different methods.

**Theorem 1.1** (Klainerman [14], Christodoulou [3]). *Suppose that the null condition is satisfied. Then, for any  $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^m)$ , there exists a positive constant  $\varepsilon_0$  such that the Cauchy problem (1.3)–(1.4) admits a unique global solution  $u \in C^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^m)$  for any  $\varepsilon \in (0, \varepsilon_0]$ .*

Christodoulou used the so-called conformal method which is based on Penrose's conformal compactification of Minkowski space. On the other hand, Klainerman used the vector field method and showed the above theorem by deriving some decay estimates in the original coordinates. In Klainerman's proof, he introduced vector fields

$$L_{c,j} = \frac{x_j}{c} \partial_t + ct \partial_j \quad (1 \leq j \leq 3), \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i \quad (1 \leq i < j \leq 3),$$

which are the generators of the Lorentz group, and the scaling operator

$$S = t \partial_t + x \cdot \nabla_x.$$

These vector fields play an important role in getting Klainerman's weighted  $L^1$ - $L^\infty$  estimates for wave equations (see also Hörmander [5]). In addition, using them, we can see that an extra decay factor is expected from the null forms. For example, we have

$$(1.6) \quad Q_0(v, w; c) = \frac{1}{t+r} \left\{ (\partial_t v) (Sw + cL_{c,r}w) - c \sum_{j=1}^3 (L_{c,j}v)(\partial_j w) \right. \\ \left. - c^2 (Sv)(\partial_r w) + c^2 \sum_{j \neq k} \omega_k (\Omega_{jk}v)(\partial_j w) \right\},$$

where  $r = |x|$ ,  $\omega = (\omega_1, \omega_2, \omega_3) = x/r$ ,  $\partial_r = \sum_{j=1}^3 \omega_j \partial_j$ ,  $L_{c,r} = \sum_{j=1}^3 \omega_j L_{c,j}$ , and  $\Omega_{ij} = -\Omega_{ji}$  for  $1 \leq j < i \leq 3$ .

Among the above vector fields, the Lorentz boost fields  $L_{c,j}$  depend on the propagation speed  $c$ , and they are unfavorable when we consider the multiple speed case. Thus, the vector field method without the Lorentz boost fields was developed by many authors (see Kovalyov [17, 18], Klainerman and Sideris [16], Yokoyama [25], Kubota and Yokoyama [19], Sideris and Tu [23], Sogge [24], Hidano [4], Katayama [9, 11], and Katayama and Yokoyama [13], for example). In place of (1.6), the following identity was used in the above works relating to the

null condition for the multiple speed case:

$$(1.7) \quad Q_0(v, w; c) = \frac{1}{t^2} (Sv + (ct - r)\partial_r v)(Sw - (ct + r)\partial_r w) \\ + \frac{c}{t} \{ (Sv)(\partial_r w) - (\partial_r v)(Sw) \} + \frac{c^2}{r} \sum_{j \neq k} \omega_k(\partial_j v)(\Omega_{jk} w),$$

whose variant was introduced by Hoshiga and Kubo [6]. Equation (1.7) leads to a good estimate in the region  $r > \delta t$  with some small  $\delta > 0$ , because  $r$  is equivalent to  $t + r$  in this region. Note that the operator  $S$  is still used in (1.7), and this is the only reason why  $S$  was adopted in [9, 19, 25], because these works are based on variants of  $L^\infty$ - $L^\infty$  estimates due to John [7] and Kovalyov [17], where only  $\partial_a$  and  $\Omega_{ij}$  are used (see Lemma 3.2 below).

Our aim here is to get rid of not only  $L_{c,j}$ , but also  $S$  from the estimate of the null forms, and prove Theorem 1.1 using only  $\partial_a$  and  $\Omega_{jk}$ . Though the usage of the scaling operator  $S$  has not caused any serious difficulty in the study of the Cauchy problem for nonlinear wave equations so far, we believe that it is worthwhile developing a simple approach with a smaller set of vector fields. For this purpose, we make use of the identity

$$(1.8) \quad Q_0(v, w; c) = \frac{1}{2} \{ (D_{+,c}v)(D_{-,c}w) + (D_{-,c}v)(D_{+,c}w) \} \\ + \frac{c^2}{r} \sum_{j \neq k} \omega_k(\partial_j v)(\Omega_{jk} w),$$

where  $D_{\pm,c} = \partial_t \pm c\partial_r$ . Note that this identity was already used implicitly to obtain identities like (1.7) (see [23], for example). In view of (1.8), what we need to treat the null forms is an enhanced decay estimate for the tangential derivative  $D_{+,c}$  to the light cone. We can say that, in the previous works, this enhanced decay has been observed through

$$D_{+,c} = \frac{1}{t} (S + (ct - r)\partial_r) \text{ or } D_{+,c} = \frac{1}{ct + r} (cS + cL_{c,r})$$

with the help of  $S$  or also  $L_{c,r} = \sum_{j=1}^3 \omega_j L_{c,j}$ .

In this paper, we take a different approach. We will establish the enhanced decay of  $D_{+,c}u$  for the solution  $u$  to the wave equation directly. We formulate it as a weighted  $L^\infty$ - $L^\infty$  estimate in Theorem 2.1 below, which is our main ingredient in this paper. The point is that such an estimate can be derived by using only  $\partial_a$  and  $\Omega_{ij}$ . This type of approach to  $D_{+,c}$  goes back to the work of John [8].

## 2. THE MAIN RESULT

Before stating our result precisely, we introduce several notations. We put  $Z = \{Z_a\}_{1 \leq a \leq 7} = \{(\partial_a)_{0 \leq a \leq 3}, (\Omega_{jk})_{1 \leq j < k \leq 3}\}$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_7)$ , we define  $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} \dots Z_7^{\alpha_7}$ . For a function  $v = v(t, x)$  and a nonnegative integer  $s$ , we define

$$(2.1) \quad |v(t, x)|_s = \sum_{|\alpha| \leq s} |Z^\alpha v(t, x)| \text{ and } \|v(t, \cdot)\|_s = \left\| |v(t, \cdot)|_s \right\|_{L^2(\mathbb{R}^3)}.$$

We put  $\langle a \rangle = \sqrt{1 + a^2}$  for  $a \in \mathbb{R}$ . Let  $c$  be a positive constant, and we fix arbitrary positive constants  $c_j$  ( $1 \leq j \leq N$ ) (our theorem is true for any choice

of these constants  $c_j$ , but when we apply our estimate to nonlinear problems, we usually choose  $c_j$  as the propagation speeds and  $N$  as the number of different propagation speeds in the system;  $c$  is also chosen from these propagation speeds). We define

$$(2.2) \quad w(t, r) = w(t, r; c_1, \dots, c_N) = \min_{0 \leq j \leq N} \langle c_j t - r \rangle$$

with  $c_0 = 0$ , and we define

$$(2.3) \quad A_{\rho, \mu, s}[G; c](t, x) = \sup_{(\tau, y) \in \Lambda_c(t, x)} |y| \langle \tau + |y| \rangle^\rho w(\tau, |y|)^{1+\mu} |G(\tau, y)|_s$$

for  $\rho, \mu \geq 0$ , a nonnegative integer  $s$ , and a smooth function  $G = G(t, x)$ , where  $\Lambda_c(t, x) = \{(\tau, y) \in [0, t] \times \mathbb{R}^3; |y - x| \leq c(t - \tau)\}$ . We also define

$$(2.4) \quad B_{\rho, s}[\phi, \psi; c](t, x) = \sup_{y \in \Lambda'_c(t, x)} \langle |y| \rangle^\rho (|\phi(y)|_{s+1} + |\psi(y)|_s)$$

for  $\rho \geq 0$ , a nonnegative integer  $s$ , and smooth functions  $\phi$  and  $\psi$  on  $\mathbb{R}^3$ , where  $\Lambda'_c(t, x) = \{y \in \mathbb{R}^3; |y - x| \leq ct\}$ .

The following theorem is our main result.

**Theorem 2.1.** *Assume  $1 \leq \kappa \leq 2$  and  $\mu > 0$ .*

(i) *Let  $u$  be the solution to*

$$\square_c u = G \quad \text{in } (0, \infty) \times \mathbb{R}^3$$

*with initial data  $u = \partial_t u = 0$  at  $t = 0$ . Then there exists a positive constant  $C$ , depending on  $\kappa$  and  $\mu$ , such that*

$$(2.5) \quad \begin{aligned} \langle |x| \rangle \langle t + |x| \rangle \langle ct - |x| \rangle^{\kappa-1} \{\log(2 + t + |x|)\}^{-1} |D_{+,c} u(t, x)| \\ \leq C A_{\kappa, \mu, 2}[G; c](t, x) \end{aligned}$$

*for  $(t, x) \in (0, \infty) \times \mathbb{R}^3$  with  $x \neq 0$ , where  $A_{\kappa, \mu, 2}$  is given by (2.3).*

*Moreover, if  $1 < \kappa < 2$ , then for any  $\delta > 0$ , there exists a constant  $C$ , depending on  $\kappa$ ,  $\mu$ , and  $\delta$ , such that*

$$(2.6) \quad \langle t + |x| \rangle^2 \langle ct - |x| \rangle^{\kappa-1} |D_{+,c} u(t, x)| \leq C A_{\kappa, \mu, 2}[G; c](t, x)$$

*for  $(t, x) \in (0, \infty) \times \mathbb{R}^3$  satisfying  $|x| > \delta t$ .*

(ii) *Let  $u^*$  be the solution to*

$$\square_c u^* = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^3$$

*with initial data  $u^* = \phi$  and  $\partial_t u^* = \psi$  at  $t = 0$ . Then we have*

$$(2.7) \quad \langle |x| \rangle \langle t + |x| \rangle \langle ct - |x| \rangle^{\kappa-1} |D_{+,c} u^*(t, x)| \leq C B_{\kappa+\mu+1, 2}[\phi, \psi; c](t, x)$$

*for  $(t, x) \in (0, \infty) \times \mathbb{R}^3$  with  $x \neq 0$ , where  $B_{\kappa+\mu+1, 2}$  is given by (2.4).*

*Remark.* (1) Similar estimates for radially symmetric solutions are obtained by Katayama [11].

(2) Suppose that  $A_{\kappa, \mu, 2}[G; c](t, x)$  is bounded on  $[0, \infty) \times \mathbb{R}^3$  for some  $\kappa \in [1, 2)$  and  $\mu > 0$  and that  $u$  solves  $\square_c u = G$  with zero initial data. Then, from Lemma 3.2 below, we see that  $u$  and  $\partial u$  decay like  $\langle t \rangle^{-1} \Psi_{\kappa-1}(t)$  along the light cone  $ct = |x|$ , where  $\Psi_\rho(t) = \log(2 + t)$  if  $\rho = 0$ , and  $\Psi_\rho(t) = 1$  if  $\rho > 0$ . Compared with this decay rate, we find from (2.5) and (2.6) that  $D_{+,c} u$  gains extra decay of  $\langle t \rangle^{-1}$  and behaves like  $\langle t \rangle^{-2} \Psi_{\kappa-1}(t)$  along the light cone.

(3) For tangential derivatives  $T_{c,j} = (x_j/|x|)\partial_t + c\partial_j$  ( $1 \leq j \leq 3$ ), Alinhac showed that

$$\left( \int_0^t \int_{\mathbb{R}^3} (1 + |c\tau - |x||)^{-\rho} |T_{c,j}u(\tau, x)|^2 dx d\tau \right)^{1/2}$$

with  $\rho > 1$  is bounded by  $\|\partial u(0, \cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|\square_c u(\tau, \cdot)\|_{L^2(\mathbb{R}^3)} d\tau$  (see [1], for example). Observe that  $T_{c,j}$  is closely connected to  $D_{+,c}$ . In fact, we have  $D_{+,c} = \sum_{j=1}^3 (x_j/|x|)T_{c,j}$ . Though Alinhac's estimate does not need  $S$  and means enhanced decay of tangential derivatives implicitly, it seems difficult to recover a pointwise decay estimate from his weighted space-time estimate. On the other hand, Sideris and Thomases [22] obtained the estimate for  $\|(1 + |ct + |\cdot||)T_{c,j}u(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ ; however,  $S$  is used in their estimate.

(4) The exterior problem for systems of nonlinear wave equations with the single or multiple speed(s) is also widely studied (see Metcalfe, Nakamura, and Sogge [20] and Metcalfe and Sogge [21] and the references cited therein). In the exterior domains, because of their unbounded coefficients on the boundary, the Lorentz boosts are unlikely to be applicable even for the single speed case. This is another reason why the vector field method without the Lorentz boosts is widely studied. In addition,  $S$  also causes a technical difficulty in the exterior problems. We will discuss the exterior problem in a subsequent paper, and we will not go into further details here.

We will prove Theorem 2.1 in the next section, after stating some known weighted  $L^\infty$ - $L^\infty$  estimates for wave equations. Though we can apply our theorem to exclude  $S$  from the proof of the multiple speed version of Theorem 1.1 in [9, 19, 25], we concentrate on the single speed case for simplicity, and we will give a new proof, without using  $S$  and  $L_{c,j}$ , of Theorem 1.1 in section 4 as an application of our main theorem.

Throughout this paper, various positive constants, which may change line by line, are denoted just by the same letter  $C$ .

### 3. PROOF OF THEOREM 2.1

For  $c > 0$ ,  $\phi = \phi(x)$ , and  $\psi = \psi(x)$ , we write  $U_c^*[\phi, \psi]$  for the solution  $u$  to the homogeneous wave equation  $\square_c u = 0$  in  $(0, \infty) \times \mathbb{R}^3$  with initial data  $u = \phi$  and  $\partial_t u = \psi$  at  $t = 0$ . Similarly, for  $c > 0$  and  $G = G(t, x)$ , we write  $U_c[G]$  for the solution  $u$  to the inhomogeneous wave equation  $\square_c u = G$  in  $(0, \infty) \times \mathbb{R}^3$  with initial data  $u = \partial_t u = 0$  at  $t = 0$ .

For  $U_c^*[\phi, \psi]$  we have the following.

**Lemma 3.1.** *Let  $c > 0$ . Then, for  $\kappa > 1$ , we have*

$$(3.1) \quad \begin{aligned} & \langle t + |x| \rangle \langle ct - |x| \rangle^{\kappa-1} |U_c^*[\phi, \psi](t, x)| \\ & \leq C \sup_{y \in \Lambda'_c(t, x)} \langle |y| \rangle^\kappa (\langle |y| \rangle |\phi(y)|_1 + |y| |\psi(y)|) \end{aligned}$$

for  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ .

For the proof, see Katayama and Yokoyama [13, Lemma 3.1] (see also Asakura [2] and Kubota and Yokoyama [19]).

After the pioneering work of John [7], a wide variety of weighted  $L^\infty$ - $L^\infty$  estimates for  $U_c[G]$  and  $\partial U_c[G]$  have been obtained (see [2, 9, 10, 12, 13, 17, 18, 19, 25]).

Here we restrict our attention to what will be used directly in our proofs of Theorems 1.1 and 2.1.

**Lemma 3.2.** *Let  $c > 0$ . Define*

$$(3.2) \quad \Phi_\rho(t, r) = \begin{cases} \log(2 + \langle t + r \rangle \langle t - r \rangle^{-1}) & \text{if } \rho = 0, \\ \langle t - r \rangle^{-\rho} & \text{if } \rho > 0, \end{cases}$$

$$(3.3) \quad \Psi_\rho(t) = \begin{cases} \log(2 + t) & \text{if } \rho = 0, \\ 1 & \text{if } \rho > 0. \end{cases}$$

Assume  $\kappa \geq 1$  and  $\mu > 0$ . Then we have

$$(3.4) \quad \langle t + |x| \rangle \Phi_{\kappa-1}(ct, |x|)^{-1} |U_c[G](t, x)| \leq CA_{\kappa, \mu, 0}[G; c](t, x),$$

$$(3.5) \quad \langle |x| \rangle \langle ct - |x| \rangle^\kappa \Psi_{\kappa-1}(t)^{-1} |\partial U_c[G](t, x)| \leq CA_{\kappa, \mu, 1}[G; c](t, x)$$

for  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ , where  $A_{\kappa, \mu, s}[G; c]$  is given by (2.3).

*Proof.* For the proof of (3.4), see Katayama and Yokoyama [13, equation (3.6) in Lemma 3.2, and section 8] for  $\kappa > 1$  and Katayama [11] for  $\kappa = 1$ .

Next we consider (3.5) with  $\kappa > 1$ . From Lemma 8.2 in [13], we find that (3.5) with  $\partial U_c[G]$  replaced by  $U_c[\partial G]$  is true. Now (3.5) follows immediately from Lemma 3.1, because we have  $\partial_a U_c[G] = U_c[\partial_a G] + \delta_{a0} U_c^*[0, G(0, \cdot)]$  for  $0 \leq a \leq 3$  with the Kronecker delta  $\delta_{ab}$ , and  $\langle |y| \rangle^{\kappa+1} |y| |G(0, y)| \leq CA_{\kappa, \mu, 1}[G; c](t)$  (note that we have  $w(0, r) = \langle r \rangle$ ). Equation (3.5) for the case  $\kappa = 1$  can be treated similarly (see [19] and [9]).  $\square$

Note that we will use (3.5) in the proof of Theorem 1.1 but not in that of Theorem 2.1.

Now we are in a position to prove Theorem 2.1. Suppose that all the assumptions in Theorem 2.1 are fulfilled. Without loss of generality, we may assume  $c = 1$ .

For simplicity of exposition, we write  $D_\pm$  for  $D_{\pm, 1} = \partial_t \pm \partial_r$ . Similarly,  $U^*[\phi, \psi]$ ,  $U[G]$ ,  $A_{\rho, \mu, s}(t, x)$ , and  $B_{\rho, s}(t, x)$  denote  $U_1^*[\phi, \psi]$ ,  $U_1[G]$ ,  $A_{\rho, \mu, s}[G; 1](t, x)$ , and  $B_{\rho, s}[\phi, \psi; 1](t, x)$ , respectively.

First we prove (2.5). Assume  $0 < r = |x| \leq 1$ . We have

$$|D_+ u| \leq |\partial_t u| + |\nabla_x u| \leq \sum_{0 \leq a \leq 3} |U[\partial_a G]| + |U^*[0, G(0, \cdot)]|.$$

From (3.4) in Lemma 3.2, we get

$$(3.6) \quad \langle t + r \rangle \Phi_{\kappa-1}(t, r)^{-1} |U[\partial_a G](t, x)| \leq CA_{\kappa, \mu, 1}(t, x),$$

while Lemma 3.1 leads to

$$\begin{aligned} \langle t + r \rangle \langle t - r \rangle^\kappa |U^*[0, G(0, \cdot)](t, x)| &\leq C \sup_{y \in \Lambda'_1(t, x)} |y| \langle |y| \rangle^{\kappa+1} |G(0, y)| \\ &\leq CA_{\kappa, \mu, 0}(t, x). \end{aligned}$$

Thus we obtain (2.5) for  $0 < |x| \leq 1$ .

We set  $v(t, r, \omega) = ru(t, r\omega)$  for  $r > 0$  and  $\omega \in S^2$ . Then we have

$$(3.7) \quad D_- D_+ v(t, r, \omega) = rG(t, r\omega) + \frac{1}{r} \sum_{1 \leq j < k \leq 3} \Omega_{jk}^2 u(t, r\omega).$$

Let  $r = |x| \geq 1$  and  $1 \leq \kappa \leq 2$ . From (3.4), we get

$$(3.8) \quad \frac{1}{r} \sum_{1 \leq j < k \leq 3} |\Omega_{jk}^2 u(t, r\omega)| \leq C \langle r \rangle^{-1} \langle t+r \rangle^{-1} \Phi_{\kappa-1}(t, r) A_{\kappa, \mu, 2}(t, r\omega) \\ \leq C \langle t+r \rangle^{-\kappa} (\langle r \rangle^{-1} + \langle t-r \rangle^{-1}) A_{\kappa, \mu, 2}(t, r\omega),$$

where  $\Phi_{\kappa-1}$  is from (3.2). It is easy to see that

$$(3.9) \quad |rG(t, r\omega)| \leq \langle t+r \rangle^{-\kappa} w(t, r)^{-1-\mu} A_{\kappa, \mu, 0}(t, r\omega).$$

Note that we have

$$A_{\kappa, \mu, s}(\tau, (t+r-\tau)\omega) \leq A_{\kappa, \mu, s}(t, r\omega) \quad \text{for } 0 \leq \tau \leq t.$$

Therefore, by (3.7), (3.8), and (3.9), we get

$$(3.10) \quad |D_+ v(t, r, \omega)| = \left| \int_0^t \frac{d}{d\tau} (D_+ v)(\tau, t+r-\tau, \omega) d\tau \right| \\ = \left| \int_0^t (D_- D_+ v)(\tau, t+r-\tau, \omega) d\tau \right| \\ \leq C \langle t+r \rangle^{-\kappa} A_{\kappa, \mu, 2}(t, r\omega) \int_0^t \langle t+r-\tau \rangle^{-1} d\tau \\ + C \langle t+r \rangle^{-\kappa} A_{\kappa, \mu, 2}(t, r\omega) \int_0^t \langle t+r-2\tau \rangle^{-1} d\tau \\ + C \langle t+r \rangle^{-\kappa} A_{\kappa, \mu, 0}(t, r\omega) \int_0^t w(\tau, t+r-\tau)^{-1-\mu} d\tau \\ \leq C \langle t+r \rangle^{-\kappa} A_{\kappa, \mu, 2}(t, r\omega) \log(2+t+r).$$

Since we have

$$rD_+ u(t, r\omega) = D_+ v(t, r, \omega) - u(t, r\omega),$$

from (3.10) and (3.4), we obtain

$$\langle r \rangle \langle t+r \rangle \langle t-r \rangle^{\kappa-1} |D_+ u(t, x)| \leq C \log(2+t+|x|) A_{\kappa, \mu, 2}(t, x)$$

for  $r = |x| \geq 1$ . This completes the proof of (2.5).

To prove (2.6), we first note that  $\langle t+r \rangle \leq C \langle r \rangle$  for  $r > \delta t$ . Let  $1 < \kappa < 2$ . By the first line of (3.8), we have

$$(3.11) \quad \frac{1}{r} \sum_{1 \leq j < k \leq 3} |\Omega_{jk}^2 u(t, r\omega)| \leq C \langle t+r \rangle^{-2} \langle t-r \rangle^{-\kappa+1} A_{\kappa, \mu, 2}(t, r\omega)$$

for  $r > \max\{\delta t, 1\}$ . Obviously  $r > \max\{\delta t, 1\}$  yields  $t+r-\tau > \max\{\delta \tau, 1\}$  for  $0 \leq \tau \leq t$ . Hence following similar lines to (3.10), we obtain

$$|D_+ v(t, r, \omega)| \leq C \langle t+r \rangle^{-\kappa} A_{\kappa, \mu, 2}(t, r\omega) \quad \text{for } r \geq \max\{\delta t, 1\}.$$

This immediately implies (2.6), because we already know that  $|D_+ u|$  (resp.,  $|D_+ u - r^{-1} D_+ v|$ ) has the desired bound for  $(\delta t <) r \leq 1$  (resp.,  $r \geq \max\{\delta t, 1\}$ ).

Now we are going to prove (2.7). Lemma 3.1 immediately implies

$$\langle t+|x| \rangle \langle t-|x| \rangle^{\kappa+\mu-1} |D_+ u^*(t, x)| \leq C B_{\kappa+\mu+1, 1}(t, x),$$

which is better than (2.7) for  $0 < |x| \leq 1$ . Lemma 3.1 also implies

$$\begin{aligned}
 (3.12) \quad & \frac{1}{r} \sum_{1 \leq j < k \leq 3} |\Omega_{jk}^2 u^*(t, x)| \\
 & \leq C \langle r \rangle^{-1} \langle t+r \rangle^{-1} \langle t-r \rangle^{1-\kappa-\mu} B_{\kappa+\mu+1,2}(t, x) \\
 & \leq C \langle t+r \rangle^{-\kappa} (\langle r \rangle^{-1-\mu} + \langle t-r \rangle^{-1-\mu}) B_{\kappa+\mu+1,2}(t, x)
 \end{aligned}$$

for  $r = |x| \geq 1$ . Set  $v^*(t, r, \omega) = ru^*(t, r\omega)$  for  $r \geq 0$  and  $\omega \in S^2$ . For  $r \geq 1$ , similarly to (3.10), we get

$$\begin{aligned}
 |D_+ v^*(t, r, \omega)| &= \left| (D_+ v^*)(0, t+r, \omega) + \int_0^t (D_- D_+ v^*)(\tau, t+r-\tau, \omega) d\tau \right| \\
 &\leq C \langle t+r \rangle^{-\kappa} B_{\kappa+1,0}(t, r\omega) \\
 &\quad + C \langle t+r \rangle^{-\kappa} B_{\kappa+\mu+1,2}(t, r\omega) \int_0^t \langle t+r-\tau \rangle^{-1-\mu} d\tau \\
 &\quad + C \langle t+r \rangle^{-\kappa} B_{\kappa+\mu+1,2}(t, r\omega) \int_0^t \langle t+r-2\tau \rangle^{-1-\mu} d\tau \\
 &\leq C \langle t+r \rangle^{-\kappa} B_{\kappa+\mu+1,2}(t, r\omega),
 \end{aligned}$$

which ends up with

$$\langle r \rangle \langle t+r \rangle \langle t-r \rangle^{\kappa-1} |D_+ u^*(t, x)| \leq C B_{\kappa+\mu+1,2}(t, x)$$

for  $r = |x| \geq 1$ . This completes the proof of (2.7).  $\square$

#### 4. PROOF OF THEOREM 1.1

As an application of Theorem 2.1, we give a new proof of Theorem 1.1. First we derive estimates for the null forms.

**Lemma 4.1.** *Let  $c$  be a positive constant, and  $v = (v_1, \dots, v_M)$ . Suppose that  $Q$  is one of the null forms. Then, for a nonnegative integer  $s$ , there exists a positive constant  $C_s$ , depending only on  $c$  and  $s$ , such that*

$$\begin{aligned}
 |Q(v_j, v_k)|_s &\leq C_s \left\{ |\partial v|_{[s/2]} \sum_{|\alpha| \leq s} |D_{+,c} Z^\alpha v| + |\partial v|_s \sum_{|\alpha| \leq [s/2]} |D_{+,c} Z^\alpha v| \right. \\
 &\quad \left. + \frac{1}{r} (|\partial v|_{[s/2]} |v|_{s+1} + |v|_{[s/2]+1} |\partial v|_s) \right\}.
 \end{aligned}$$

*Proof.* The case  $Q = Q_0$  and  $s = 0$  follows immediately from (1.8). We can obtain similar identities for other null forms by using

$$(\partial_t, \nabla_x) = \left( \frac{1}{2}, -\frac{x}{2cr} \right) D_{-,c} + \left( \frac{1}{2}, \frac{x}{2cr} \right) D_{+,c} - \left( 0, \frac{x}{r^2} \wedge \Omega \right)$$

with  $\Omega = (\Omega_{23}, -\Omega_{13}, \Omega_{12})$  (see (5.2) in Sideris and Tu [23, Lemma 5.1]), and we can show the desired estimate for  $s = 0$ . Since  $Z^\alpha Q(v_j, v_k)$  can be written in terms of  $Q_0(Z^\beta v_j, Z^\gamma v_k; c)$  and  $Q_{ab}(Z^\beta v_j, Z^\gamma v_k)$  ( $0 \leq a < b \leq 3$ ) with  $|\beta| + |\gamma| \leq |\alpha|$ , the desired estimate for general  $s$  follows immediately.  $\square$



Now we are going to prove Theorem 1.1. Without loss of generality, we may assume  $c = 1$ . Assume that the assumptions in Theorem 1.1 are fulfilled. Let  $u$  be the solution to (1.3)–(1.4) on  $[0, T) \times \mathbb{R}^3$ , and we set

$$e_{\rho,k}(t, x) = \langle t + |x| \rangle \langle t - |x| \rangle^\rho |u(t, x)|_{k+2} + \langle |x| \rangle \langle t - |x| \rangle^{\rho+1} |\partial u(t, x)|_{k+1} \\ + \chi(t, x) \langle t + |x| \rangle^2 \langle t - |x| \rangle^\rho \sum_{|\alpha| \leq k} |D_{+,1} Z^\alpha u(t, x)|$$

for  $\rho > 0$  and a positive integer  $k$ , where  $\chi(t, x) = 1$  if  $|x| > (1+t)/2$ , while  $\chi(t, x) = 0$  if  $|x| \leq (1+t)/2$ . We fix  $\rho \in (1/2, 1)$  and  $s \geq 8$ , and assume that

$$(4.1) \quad \sup_{0 \leq t < T} \|e_{\rho,s}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq M\varepsilon$$

holds for some large  $M(> 0)$  and small  $\varepsilon(> 0)$ , satisfying  $M\varepsilon \leq 1$ . Our goal here is to get (4.1) with  $M$  replaced by  $M/2$ . Once such an estimate is established, it is well known that we can obtain Theorem 1.1 by the so-called bootstrap (or continuity) argument.

In the following we always assume  $M$  is large enough, and  $\varepsilon$  is sufficiently small. For simplicity of exposition, we will not write dependence of nonlinearities on the unknowns explicitly. Namely we abbreviate  $F(u, \partial u, \nabla_x \partial u)(t, x)$  as  $F(t, x)$ , and so on.

First we evaluate the energy. For any nonnegative integer  $k \leq 2s$ , (4.1) implies

$$(4.2) \quad |F^{(2)}(t, x)|_k \leq CM\varepsilon \langle |x| \rangle^{-1} \langle t - |x| \rangle^{-1-\rho} |\partial u(t, x)|_{k+1},$$

where  $F^{(2)}$  denotes the quadratic terms of  $F$ . Put  $H = F - F^{(2)}$ , and  $Zu = (Z_1 u, \dots, Z_7 u)$ . Since we have

$$(4.3) \quad \langle r \rangle^{-1} \langle t - r \rangle^{-1} \leq C \langle t + r \rangle^{-1} \quad \text{for any } (t, r) \in [0, \infty) \times [0, \infty),$$

and since  $\langle |x| \rangle^{-1} |Zu| \leq C|\partial u|$ , from (4.1) we obtain

$$(4.4) \quad |H(t, x)|_k \leq C(|u|^3 + |(u, \partial u)|_{[k/2]+1}^2 (|Zu|_{k-1} + |\partial u|_{k+1})) \\ \leq CM^3 \varepsilon^3 \langle t + |x| \rangle^{-3} \langle t - |x| \rangle^{-3\rho} \\ + CM^2 \varepsilon^2 \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-2\rho} |\partial u(t, x)|_{k+1}$$

for any nonnegative integer  $k \leq 2s$ . Similarly to (4.2) and (4.4), using (4.3), we obtain

$$(4.5) \quad |F_{i,\alpha}(t, x)| \leq CM\varepsilon(1+t)^{-1} |\partial u(t, x)|_{2s} + CM^3 \varepsilon^3 \langle t + |x| \rangle^{-3} \langle t - |x| \rangle^{-3\rho}$$

for  $|\alpha| \leq 2s$ , where

$$F_{i,\alpha} = Z^\alpha F_i - \sum_{j,k,a} c_{ka}^{ij} \partial_k \partial_a (Z^\alpha u_j)$$

with  $c_{ka}^{ij}$  coming from (1.5). It is easy to see that

$$(4.6) \quad \|\langle t + |\cdot| \rangle^{-3} \langle t - |\cdot| \rangle^{-3\rho}\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-2}$$

for  $\rho > 1/2$ . Therefore, from (4.5), we obtain

$$\|F_{i,\alpha}(t, \cdot)\|_{L^2} \leq CM\varepsilon(1+t)^{-1} \|\partial u(t, \cdot)\|_{2s} + CM^3 \varepsilon^3 (1+t)^{-2}$$

for  $|\alpha| \leq 2s$ . We also have

$$\sum_{j,k,a} |c_{ka}^{ij}(t, x)|_1 \leq CM\varepsilon(1+t)^{-1}.$$

Now, applying the energy inequality for the systems of perturbed wave equations  $\square_1(Z^\alpha u_i) - \sum_{j,k,a} c_{ka}^{ij} \partial_k \partial_a (Z^\alpha u_j) = F_{i,\alpha}$ , we find

$$\frac{d}{dt} \|\partial u(t, \cdot)\|_{2s} \leq CM\varepsilon(1+t)^{-1} \|\partial u(t, \cdot)\|_{2s} + CM^3\varepsilon^3(1+t)^{-2},$$

and the Gronwall lemma leads to

$$(4.7) \quad \|\partial u(t, \cdot)\|_{2s} \leq C(\varepsilon + M^3\varepsilon^3)(1+t)^{C_0M\varepsilon} \leq CM\varepsilon(1+t)^{C_0M\varepsilon}$$

with an appropriate positive constant  $C_0$  which is independent of  $M$  (note that the energy inequality for the systems of perturbed wave equations is available because of the symmetry condition).

In the following, we repeatedly use Theorem 2.1 and Lemmas 3.1 and 3.2 with the choice of  $N = 1$  and  $c_1 = 1(= c)$ . In other words, from now on we put  $w(t, r) = \min\{\langle r \rangle, \langle t - r \rangle\}$ . Note that we have

$$(4.8) \quad \langle r \rangle^{-1} \langle t - r \rangle^{-1} \leq C \langle t + r \rangle^{-1} w(t, r)^{-1},$$

which is more precise than (4.3).

By (4.7) and the Sobolev-type inequality

$$\langle |x| \rangle |v(t, x)| \leq C \|v(t, \cdot)\|_2,$$

whose proof can be found in Klainerman [15], we see that

$$(4.9) \quad \langle |x| \rangle |\partial u(t, x)|_{2s-2} \leq CM\varepsilon(1+t)^{C_0M\varepsilon}.$$

Using (4.8) and (4.9), from (4.2) and (4.4) with  $k = 2s - 3$ , we obtain

$$|F(t, x)|_{2s-3} \leq CM^2\varepsilon^2 \langle r \rangle^{-1} \langle t + |x| \rangle^{-1} w(t, |x|)^{-2\rho} (1+t)^{C_0M\varepsilon},$$

which implies

$$(4.10) \quad A_{1+\nu, 2\rho-1, 2s-3}[F; 1](t, x) \leq CM^2\varepsilon^2 \langle t + |x| \rangle^{C_0M\varepsilon+\nu},$$

where  $\nu$  is a positive constant to be fixed later (note that we have  $\langle \tau + |y| \rangle \leq \langle t + |x| \rangle$  for  $(\tau, y) \in \Lambda_1(t, x)$ ). Since  $2\rho > 1$  and  $1 + \nu > 1$ , by Lemmas 3.1 and 3.2 with Theorem 2.1, we obtain

$$(4.11) \quad \begin{aligned} e_{0, 2s-5}(t, x) &\leq e_{\nu, 2s-5}(t, x) \leq C\varepsilon + CM^2\varepsilon^2 \langle t + |x| \rangle^{C_0M\varepsilon+\nu} \\ &\leq CM\varepsilon \langle t + |x| \rangle^{C_0M\varepsilon+\nu}. \end{aligned}$$

Finally, we are going to estimate  $e_{\rho, s}(t, x)$ . By (4.11) and (4.2) with  $k = 2s - 6$ , we have

$$|F^{(2)}(t, x)|_{2s-6} \leq CM^2\varepsilon^2 \langle t + |x| \rangle^{-2-\rho+C_0M\varepsilon+\nu} \langle |x| \rangle^{-2}$$

for  $(t, x)$  satisfying  $|x| \leq (t+1)/2$ . On the other hand, (4.1), (4.11), and Lemma 4.1 imply

$$|F^{(2)}(t, x)|_{2s-6} \leq CM^2\varepsilon^2 \langle t + |x| \rangle^{-3+C_0M\varepsilon+\nu} \langle t - |x| \rangle^{-1-\rho}$$

for  $(t, x)$  satisfying  $|x| \geq (t+1)/2$ . Summing up, we obtain

$$(4.12) \quad |F^{(2)}(t, x)|_{2s-6} \leq CM^2\varepsilon^2 \langle |x| \rangle^{-1} \langle t + |x| \rangle^{-2+C_0M\varepsilon+\nu} w(t, |x|)^{-1-\rho}.$$

By the first line of (4.4) with  $k = 2s - 6$ , using (4.1) and (4.11), we get

$$(4.13) \quad |H(t, x)|_{2s-6} \leq CM^3\varepsilon^3 \langle |x| \rangle^{-1} \langle t + |x| \rangle^{-2+C_0M\varepsilon+\nu} w(t, |x|)^{-2\rho}.$$

Equations (4.12) and (4.13) yield

$$(4.14) \quad |F(t, x)|_{2s-6} \leq CM^2\varepsilon^2 \langle |x| \rangle^{-1} \langle t + |x| \rangle^{-2+C_0M\varepsilon+\nu} w(t, |x|)^{-2\rho}.$$

Now we fix some  $\nu$  satisfying  $0 < \nu < 1 - \rho$ , and assume that  $\varepsilon$  is sufficiently small to satisfy  $-2 + C_0 M \varepsilon + \nu \leq -1 - \rho$ . Then from (4.14) we find that

$$(4.15) \quad A_{1+\rho, 2\rho-1, 2s-6}[F; 1](t, x) \leq CM^2 \varepsilon^2.$$

Since we have  $s+2 \leq 2s-6$ ,  $1+\rho > 1$ , and  $2\rho > 1$ , from Theorem 2.1, Lemmas 3.1 and 3.2, we obtain

$$(4.16) \quad e_{\rho, s}(t, x) \leq C_1(\varepsilon + M^2 \varepsilon^2)$$

for  $(t, x) \in [0, T) \times \mathbb{R}^3$ , with an appropriate positive constant  $C_1$  which is independent of  $M$ . Finally, if  $M$  is large enough to satisfy  $4C_1 \leq M$ , and  $\varepsilon$  is small enough to satisfy  $C_1 M \varepsilon \leq 1/4$ , by (4.16) we obtain

$$(4.17) \quad \sup_{0 \leq t < T} \|e_{\rho, s}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{M}{2} \varepsilon,$$

which is the desired result. This completes the proof.

#### REFERENCES

- [1] S. ALINHAC, *Remarks on energy inequalities for wave and Maxwell equations on a curved back ground*, Math. Ann., 329 (2004), pp. 707–722.
- [2] F. ASAKURA, *Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions*, Comm. Partial Differential Equations, 11 (1986), pp. 1459–1487.
- [3] D. CHRISTODOULOU, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math., 39 (1986), pp. 267–282.
- [4] K. HIDANO, *The global existence theorem for quasi-linear wave equations with multiple speeds*, Hokkaido Math. J., 33 (2004), pp. 607–636.
- [5] L. HÖRMANDER,  *$L^1$ ,  $L^\infty$  estimates for the wave operator*, in Analyse Mathématique et Applications, Contributions en l'Honneur de J. L. Lions, Gauthier–Villars, Paris, 1988, pp. 211–234.
- [6] A. HOSHIGA AND H. KUBO, *Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition*, SIAM J. Math. Anal., 31 (2000), pp. 486–513.
- [7] F. JOHN, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., 28 (1979), pp. 235–268.
- [8] F. JOHN, *Lower bounds for the life span of solutions of nonlinear wave equations in three space dimensions*, Comm. Pure Appl. Math., 36 (1983), pp. 1–35.
- [9] S. KATAYAMA, *Global and almost-global existence for systems of nonlinear wave equations with different propagation speeds*, Differential Integral Equations, 17 (2004), pp. 1043–1078.
- [10] S. KATAYAMA, *Global existence for systems of wave equations with nonresonant nonlinearities and null forms*, J. Differential Equations, 209 (2005), pp. 140–171.
- [11] S. KATAYAMA, *Lifespan for radially symmetric solutions to systems of semilinear wave equations with multiple speeds*, Osaka J. Math, to appear.
- [12] S. KATAYAMA AND A. MATSUMURA, *Sharp lower bound for the lifespan of the systems of semilinear wave equations with multiple speeds*, J. Math. Kyoto Univ., 45 (2005), pp. 391–403.
- [13] S. KATAYAMA AND K. YOKOYAMA, *Global small amplitude solutions to systems of nonlinear wave equations with multiple speeds*, Osaka J. Math., 43 (2006), pp. 283–326.
- [14] S. KLAINERMAN, *The null condition and global existence to nonlinear wave equations*, in Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1, Lectures in Appl. Math. 23, AMS, Providence, RI, 1986, pp. 293–326.
- [15] S. KLAINERMAN, *Remarks on the global Sobolev inequalities in the Minkowski space  $\mathbf{R}^{n+1}$* , Comm. Pure Appl. Math., 40 (1987), pp. 111–117.
- [16] S. KLAINERMAN AND T. C. SIDERIS, *On almost global existence for nonrelativistic wave equations in 3D*, Comm. Pure Appl. Math., 49 (1996), pp. 307–321.
- [17] M. KOVALYOV, *Long-time behavior of solutions of a system of nonlinear wave equations*, Comm. Partial Differential Equations, 12 (1987), pp. 471–501.

- [18] M. KOVALYOV, *Resonance-type behaviour in a system of nonlinear wave equations*, J. Differential Equations, 77 (1989), pp. 73–83.
- [19] K. KUBOTA AND K. YOKOYAMA, *Global existence of classical solutions to systems of nonlinear wave equations with different speeds of propagation*, Japan. J. Math. (N.S.), 27 (2001), pp. 113–202.
- [20] J. METCALFE, M. NAKAMURA, AND C. D. SOGGE, *Global existence of quasilinear, nonrelativistic wave equations satisfying the null condition*, Japan. J. Math. (N.S.), 31 (2005), pp. 391–472.
- [21] J. METCALFE AND C. D. SOGGE, *Global existence of null-form wave equations in exterior domains*, Math. Z., 256 (2007), pp. 521–549.
- [22] T. C. SIDERIS AND B. THOMASES, *Local energy decay for solutions of multidimensional isotropic symmetric hyperbolic systems*, J. Hyperbolic Differ. Equ., 3 (2006), pp. 673–690.
- [23] T. C. SIDERIS AND S.-Y. TU, *Global existence for systems of nonlinear wave equations in 3D with multiple speeds*, SIAM J. Math. Anal., 33 (2001), pp. 477–488.
- [24] C. D. SOGGE, *Global existence for nonlinear wave equations with multiple speeds*, in Harmonic Analysis at Mount Holyoke, W. Beckner, A. Nagel, A. Seeger, and H. F. Smith, eds., Contemp. Math. 320, AMS, Providence, RI, 2003, pp. 353–366.
- [25] K. YOKOYAMA, *Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions*, J. Math. Soc. Japan, 52 (2000), pp. 609–632.

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